Chaos in disordered nonlinear lattices

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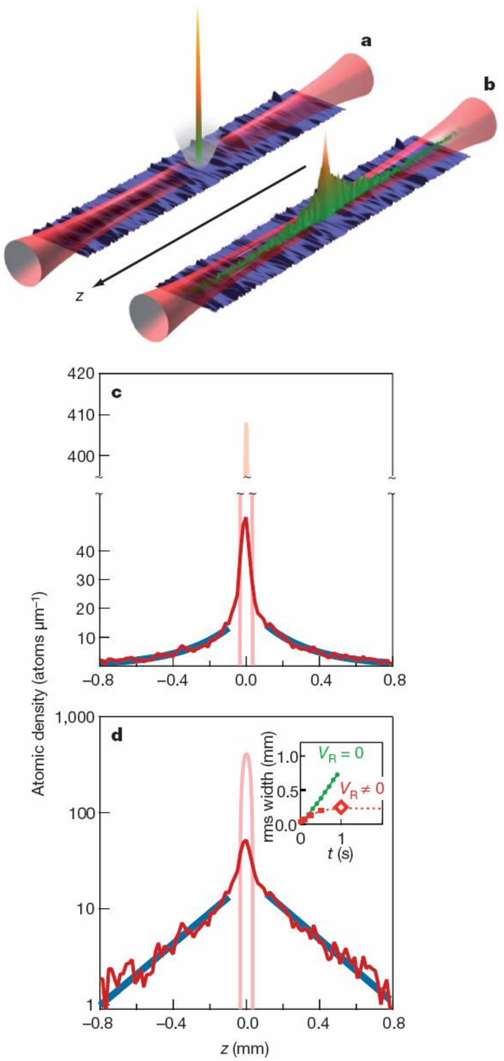
Outline

- **Dynamical Systems:**
  - ✓ **The quartic Klein-Gordon (KG) disordered lattice**
  - ✓ **The disordered nonlinear Schrödinger equation (DNLS)**
- **Numerical methods**
- **Different dynamical behaviors**
  - ✓ **Single site excitations**
  - ✓ **Block excitations**
- **Summary**

Interplay of disorder and nonlinearity

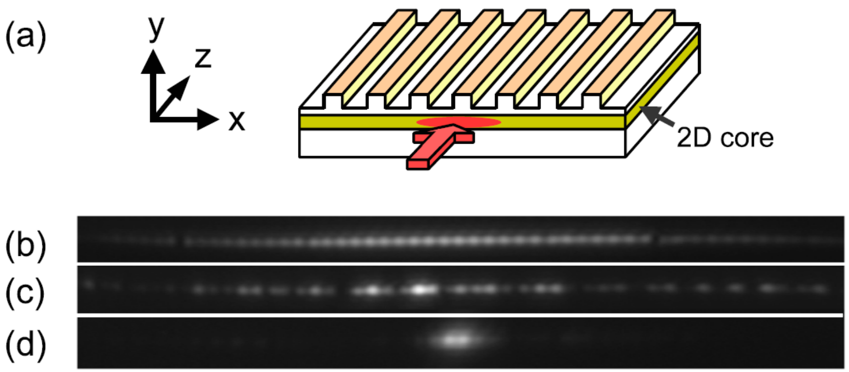
Waves in disordered media – Anderson localization
[Anderson, Phys. Rev. (1958)]. Experiments on BEC [Billy et al., Nature (2008)].



Waves in nonlinear disordered media – localization or delocalization?

Theoretical and/or numerical studies [Shepelyansky, PRL (1993) – Molina, PRB (1998) – Pikovsky & Shepelyansky, PRL (2008) – Kopidakis et al. PRL (2008)]

Experiments: propagation of light in disordered 1d waveguide lattices [Lahini et al., PRL (2008)].



The Klein – Gordon (KG) model

$$H_K = \sum_{l=1}^{N} \frac{p_l^2}{2} + \frac{\tilde{\varepsilon}_l}{2} u_l^2 + \frac{1}{4} u_l^4 + \frac{1}{2W} \left(u_{l+1} - u_l \right)^2$$

with **fixed boundary conditions** $u_0=p_0=u_{N+1}=p_{N+1}=0$. Typically N=1000.

Parameters: **W** and the **total energy E.** $\tilde{\varepsilon}_l$ **chosen uniformly from** $\left[\frac{1}{2}, \frac{3}{2}\right]$.

The discrete nonlinear Schrödinger (DNLS) equation

$$H_D = \sum_{l=1}^{N} \varepsilon_l \left|\psi_l\right|^2 + \frac{\beta}{2} \left|\psi_l\right|^4 - \left(\psi_{l+1}\psi_l^* + \psi_{l+1}^*\psi_l \right)^2, \quad \psi_l = \frac{1}{\sqrt{2}} \left(q_l + i p_l \right)$$

where ε_l **chosen uniformly from** $\left[-\frac{W}{2}, \frac{W}{2}\right]$ and β **is the nonlinear parameter.**

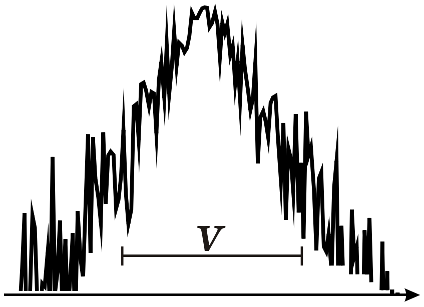
Scales

Linear case: $\omega_\nu^2 \in \left[\frac{1}{2}, \frac{3}{2} + \frac{4}{W}\right]$, **width of the squared frequency spectrum:**

$$\Delta_K = 1 + \frac{4}{W}$$

$$(\Delta_D = W + 4)$$

Localization volume of an eigenstate: $V \sim \frac{1}{\sum_{l=1}^{N} A_{\nu,l}^4}$



Average spacing of squared eigenfrequencies of NMs within the range of a localization volume: $d_K \approx \frac{\Delta_K}{V}$

Nonlinearity induced squared frequency shift of a single site oscillator

$$\delta_l = \frac{3E_l}{2\tilde{\varepsilon}_l} \propto E \qquad (\delta_l = \beta |\psi_l|^2)$$

The relation of the two scales $d_K \leq \Delta_K$ with the nonlinear frequency shift δ_l determines the packet evolution.

Distribution characterization

We consider normalized energy distributions in normal mode (NM) space

$$z_\nu \equiv \frac{E_\nu}{\sum_m E_m} \quad \text{with} \quad E_\nu = \frac{1}{2}\left(\dot{A}_\nu^2 + \omega_\nu^2 A_\nu^2\right)$$

, where A_ν is the amplitude of the νth NM.

Second moment: $$m_2 = \sum_{\nu=1}^{N} \left(\nu - \bar{\nu}\right)^2 z_\nu \quad \text{with} \quad \bar{\nu} = \sum_{\nu=1}^{N} \nu z_\nu$$

Participation number: $$P = \frac{1}{\sum_{\nu=1}^{N} z_\nu^2}$$

measures the number of stronger excited modes in z_ν. Single mode $P=1$, equipartition of energy $P=N$.

Lyapunov Exponents

Roughly speaking, the Lyapunov exponents of a given orbit characterize the mean exponential rate of divergence of trajectories surrounding it [see e.g. Ch.S., LNP, (2010)].

Consider an orbit in the 2N-dimensional phase space with initial condition x(0) and an initial deviation vector from it v(0). Then the mean exponential rate of divergence is:

$$\sigma(x(0), v(0)) = \lim_{t\to\infty} \frac{1}{t} \ln \frac{\|v(t)\|}{\|v(0)\|}$$

$\sigma_1 = 0 \rightarrow$ **Regular motion**

$\sigma_1 \neq 0 \rightarrow$ **Chaotic motion**

Computational methods

Consider an N degree of freedom autonomous Hamiltonian systems of the form:

$$H(\vec{q},\vec{p}) = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + V(\vec{q})$$

As an example, we take the Hénon-Heiles system:

$$H_2 = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}y^3$$

Hamilton equations of motion:

$$\begin{cases} \dot{x} &= p_x \\ \dot{y} &= p_y \\ \dot{p}_x &= -x - 2xy \\ \dot{p}_y &= y^2 - x^2 - y \end{cases}$$

Variational equations:

$$\begin{cases} \dot{\delta x} &= \delta p_x \\ \dot{\delta y} &= \delta p_y \\ \dot{\delta p_x} &= -(1+2y)\delta x - 2x\delta y \\ \dot{\delta p_y} &= -2x\delta x + (-1+2y)\delta y \end{cases}$$

Symplectic integration schemes

If the Hamiltonian H can be split into two integrable parts as H=A+B, a symplectic scheme for integrating the equations of motion from time t to time t+τ consists of approximating the operator $e^{\tau L_H}$, i.e. the solution of Hamilton equations of motion, by

$$e^{\tau L_H} = e^{\tau (L_A + L_B)} \approx \prod_{i=1}^{j} e^{c_i \tau L_A} e^{d_i \tau L_B}$$

for appropriate values of constants c_i, d_i.

So the dynamics over an integration time step τ is described by a series of successive acts of Hamiltonians A and B.

We consider a particular symplectic integrator [Laskar & Robutel, Cel. Mech. Dyn. Astr. (2001)].

$$SABA_2 = e^{\left[\frac{(3-\sqrt{3})}{6}\tau\right] L_A} e^{\frac{\tau}{2} L_B} e^{\frac{\sqrt{3}\tau}{3} L_A} e^{\frac{\tau}{2} L_B} e^{\left[\frac{(3-\sqrt{3})}{6}\tau\right] L_A}$$

Tangent Map (TM) Method

We use symplectic integration schemes for the integrating the equations of motion AND THE VARIATIONAL EQUATIONS.

The Hénon-Heiles system can be split as:

$$A = \frac{1}{2}(p_x^2 + p_y^2), \qquad B = \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}y^3,$$

The system of the Hamilton equations of motion and the variational equations is split into two integrable systems which correspond to Hamiltonians A and B.

$$\begin{aligned} \dot{x} &= p_x \\ \dot{y} &= p_y \\ \dot{p}_x &= -x - 2xy \\ \dot{p}_y &= y^2 - x^2 - y \\ \dot{\delta x} &= \delta p_x \\ \dot{\delta y} &= \delta p_y \\ \dot{\delta p}_x &= -(1+2y)\delta x - 2x\delta y \\ \dot{\delta p}_y &= -2x\delta x + (-1+2y)\delta y \end{aligned}$$

$A(\vec{p}) \longrightarrow$

$$e^{\tau L_{AV}} : \begin{cases} x' &= x + p_x \tau \\ y' &= y + p_y \tau \\ px' &= p_x \\ py' &= p_y \\ \delta x' &= \delta x + \delta p_x \tau \\ \delta y' &= \delta y + \delta p_y \tau \\ \delta p_x' &= \delta p_x \\ \delta p_y' &= \delta p_y \end{cases}$$

$B(\vec{q}) \longrightarrow$

$$e^{\tau L_{BV}} : \begin{cases} x' &= x \\ y' &= y \\ p_x' &= p_x - x(1+2y)\tau \\ p_y' &= p_y + (y^2 - x^2 - y)\tau \\ \delta x' &= \delta x \\ \delta y' &= \delta y \\ \delta p_x' &= \delta p_x - \left[(1+2y)\delta x + 2x\delta y\right]\tau \\ \delta p_y' &= \delta p_y + \left[-2x\delta x + (-1+2y)\delta y\right]\tau \end{cases}$$

Tangent Map (TM) Method

So any symplectic integration scheme used for solving the Hamilton equations of motion, which involves the act of Hamiltonians A and B, can be extended in order to integrate simultaneously the variational equations **[Ch. S. & Gerlach, PRE (2010) – Gerlach & Ch.S., Discr. Cont. Dyn. Sys. (2011), Gerlach , Eggl, Ch.S., IJBC (2012)].**

$$e^{\tau L_A} : \begin{cases} x' &= x + p_x \tau \\ y' &= y + p_y \tau \\ p_x' &= p_x \\ p_y' &= p_y \end{cases} \longrightarrow e^{\tau L_{AV}} : \begin{cases} x' &= x + p_x \tau \\ y' &= y + p_y \tau \\ px' &= p_x \\ py' &= p_y \\ \delta x' &= \delta x + \delta p_x \tau \\ \delta y' &= \delta y + \delta p_y \tau \\ \delta p_x' &= \delta p_x \\ \delta p_y' &= \delta p_y \end{cases}$$

$$e^{\tau L_B} : \begin{cases} x' &= x \\ y' &= y \\ p_x' &= p_x - x(1+2y)\tau \\ p_y' &= p_y + (y^2 - x^2 - y)\tau \end{cases} \longrightarrow e^{\tau L_{BV}} : \begin{cases} x' &= x \\ y' &= y \\ p_x' &= p_x - x(1+2y)\tau \\ p_y' &= p_y + (y^2 - x^2 - y)\tau \\ \delta x' &= \delta x \\ \delta y' &= \delta y \\ \delta p_x' &= \delta p_x - \left[(1+2y)\delta x + 2x\delta y\right]\tau \\ \delta p_y' &= \delta p_y + \left[-2x\delta x + (-1+2y)\delta y\right]\tau \end{cases}$$

Symplectic Integrator $SABA_2C$

The integrator

$$SABA_2 = e^{\left[\frac{(3-\sqrt{3})}{6}\tau\right]L_A} e^{\frac{\tau}{2}L_B} e^{\frac{\sqrt{3}\tau}{3}L_A} e^{\frac{\tau}{2}L_B} e^{\left[\frac{(3-\sqrt{3})}{6}\tau\right]L_A}$$

has only small positive steps and its error is of order $O(\tau^2)$.

In the case where *A* is quadratic in the momenta and *B* depends only on the positions the method can be improved by introducing a corrector *C*, having a small negative step:

$$e^{-\tau^3 \frac{c}{2} L_{\{\{A,B\},B\}}}$$

with $c = \frac{2-\sqrt{3}}{24}$.

Thus the full integrator scheme becomes: $SABAC_2 = C\,(SABA_2)\,C$ and its error is of order $O(\tau^4)$.

Symplectic Integrator SABA$_2$C for the KG system

We apply the SABAC$_2$ integrator scheme to the KG Hamiltonian by using the splitting:

$$A = \sum_{l=1}^{N} \frac{p_l^2}{2}$$

$$B = \sum_{l=1}^{N} \frac{\tilde{\varepsilon}_l}{2} u_l^2 + \frac{1}{4} u_l^4 + \frac{1}{2W} \left(u_{l+1} - u_l \right)^2$$

with a corrector term which corresponds to the Hamiltonian function:

$$C = \{\{A, B\}, B\} = \sum_{l=1}^{N} \left[u_l (\tilde{\varepsilon}_l + u_l^2) - \frac{1}{W} (u_{l-1} + u_{l+1} - 2u_l) \right]^2 .$$

Different Dynamical Regimes

Three expected evolution regimes **[Flach, Chem. Phys (2010) - Ch.S., Flach, PRE (2010) - Laptyeva et al., EPL (2010) - Bodyfelt et al., PRE(2011)]:**

Weak Chaos Regime: $\delta<d$, $m_2 \sim t^{1/3}$

Frequency shift is less than the average spacing of interacting modes. NMs are weakly interacting with each other. [Molina PRB (1998) – Pikovsky, Shepelyansky, PRL (2008)].

Intermediate Strong Chaos Regime: $d<\delta<\Delta$, $m_2 \sim t^{1/2} \longrightarrow m_2 \sim t^{1/3}$

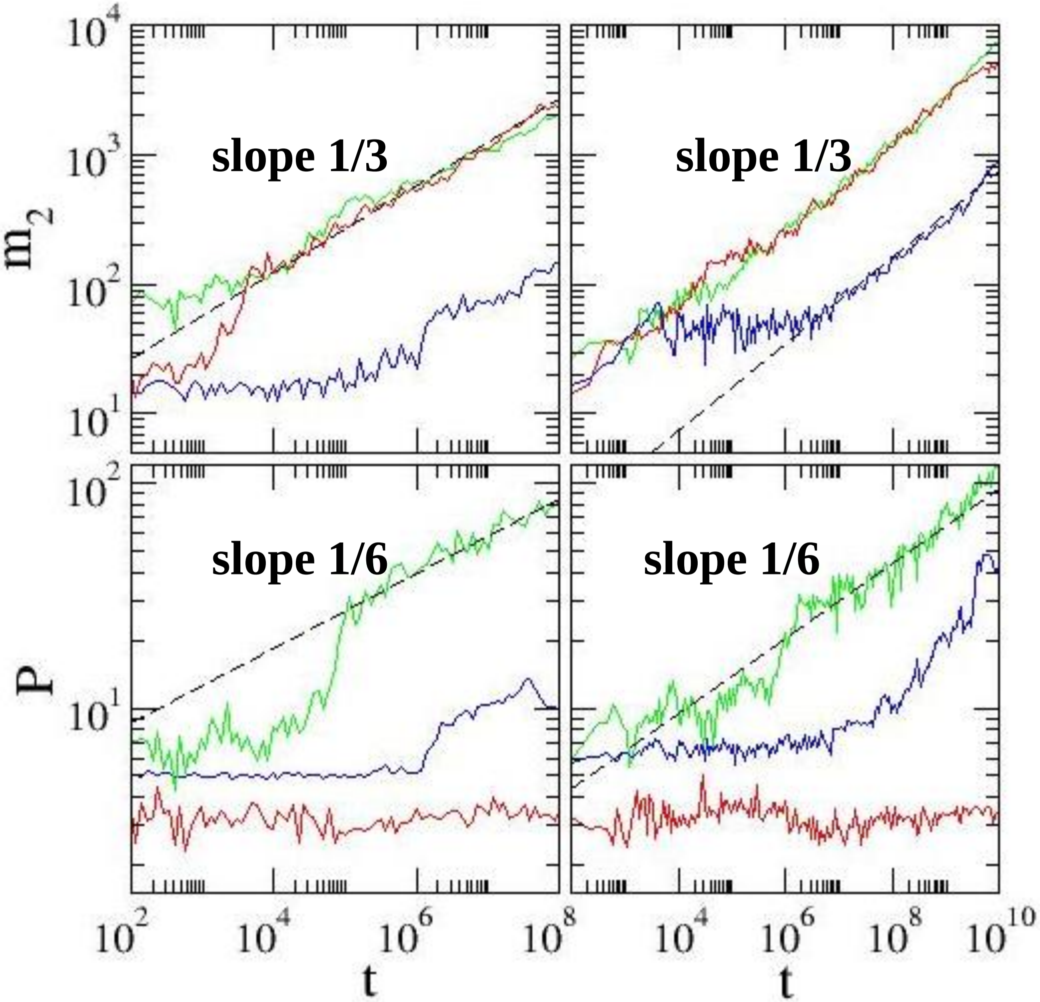
Almost all NMs in the packet are resonantly interacting. Wave packets initially spread faster and eventually enter the weak chaos regime.

Selftrapping Regime: $\delta>\Delta$

Frequency shift exceeds the spectrum width. Frequencies of excited NMs are tuned out of resonances with the nonexcited ones, leading to selftrapping, while a small part of the wave packet subdiffuses [Kopidakis et al., PRL (2008)].

Single site excitations

DNLS W=4, β= 0.1, 1, 4.5 **KG** W = 4, E = 0.05, 0.4, 1.5



No strong chaos regime

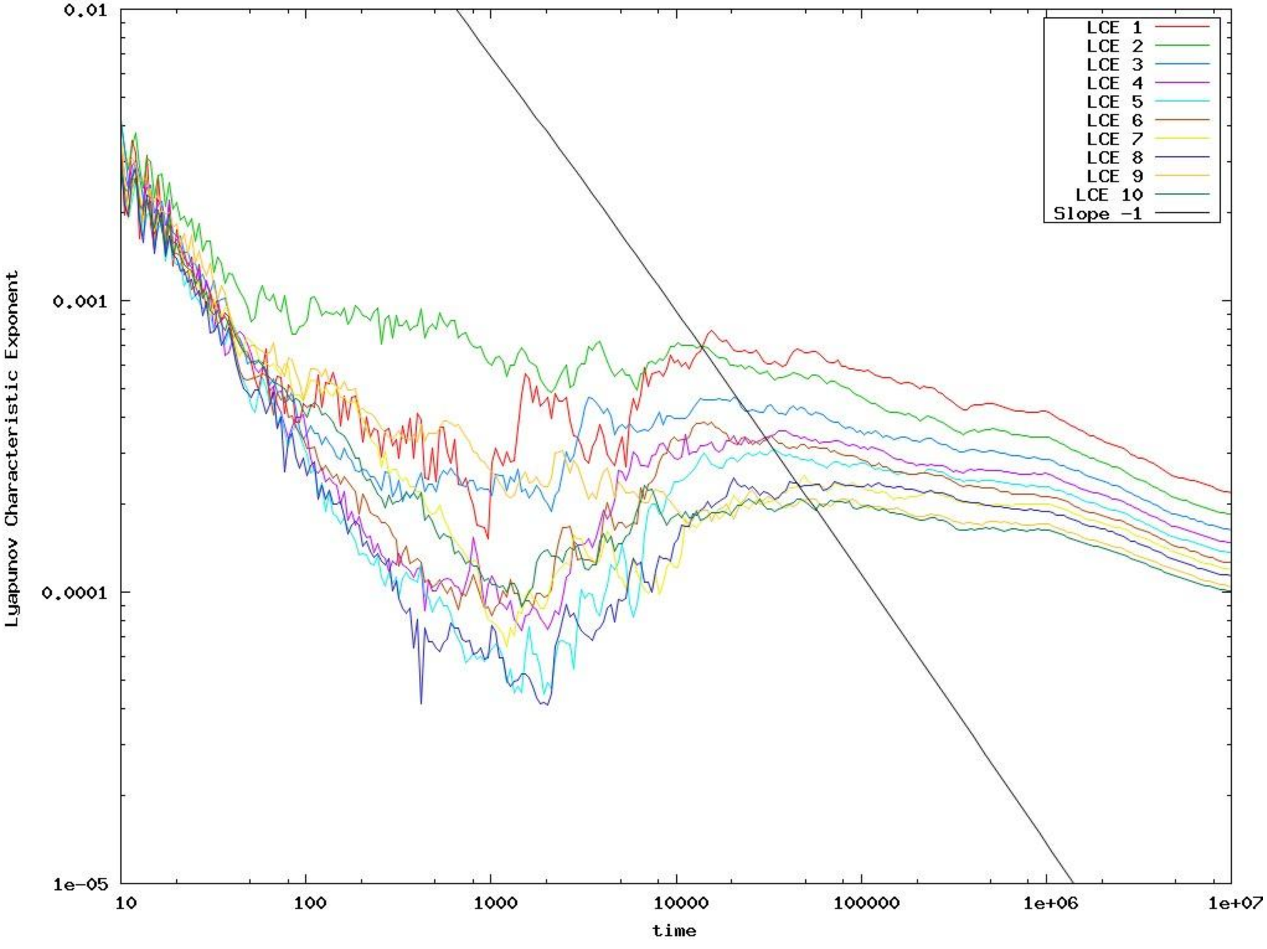
In weak chaos regime we averaged the measured exponent α ($m_2 \sim t^\alpha$) over 20 realizations:

α=0.33±0.05 (KG)

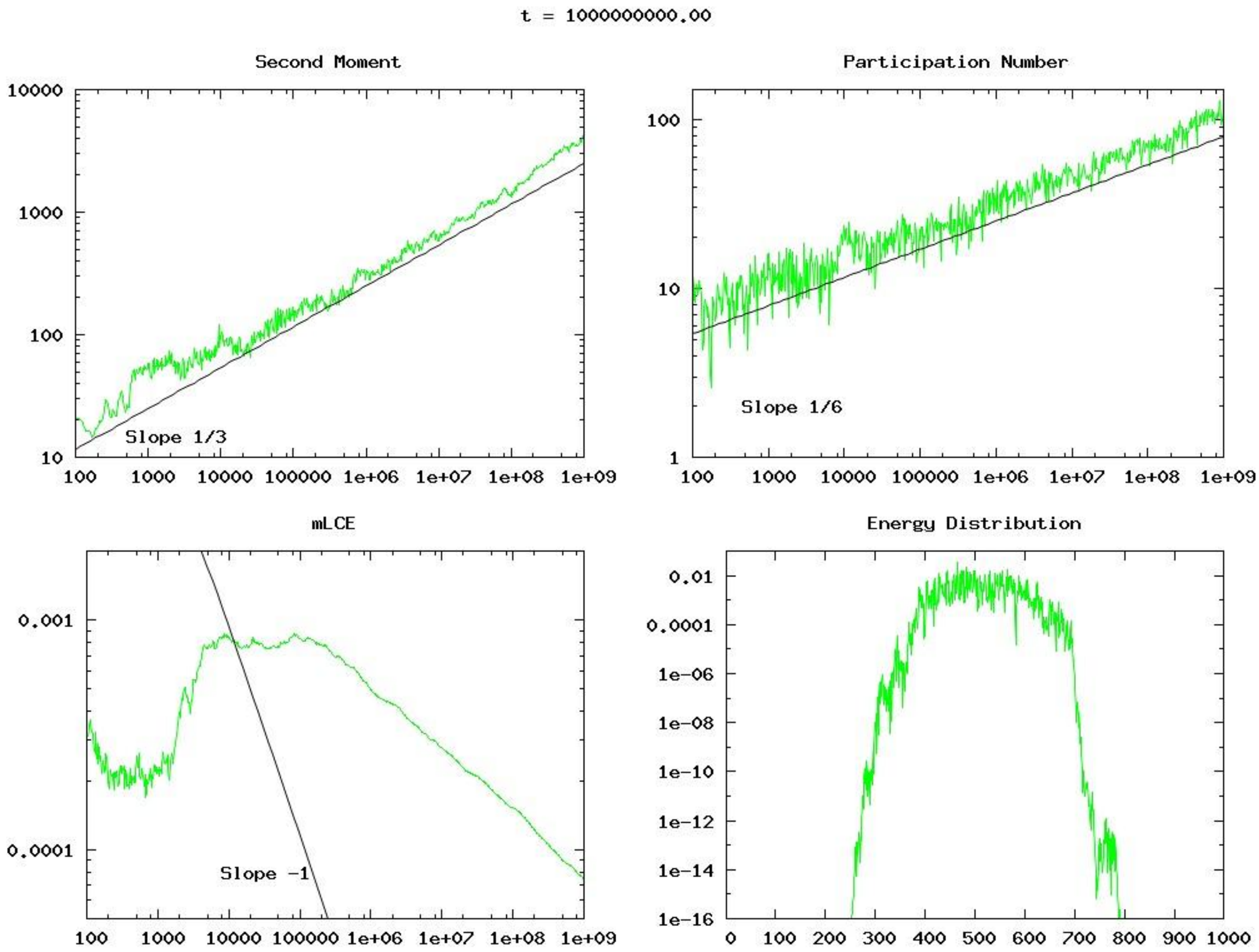
α=0.33±0.02 (DLNS)

[Flach, Krimer, Ch. S., PRL (2009) – Ch. S., Krimer, Komineas, Flach, PRE (2009) – Ch. S., Flach, PRE (2010)]

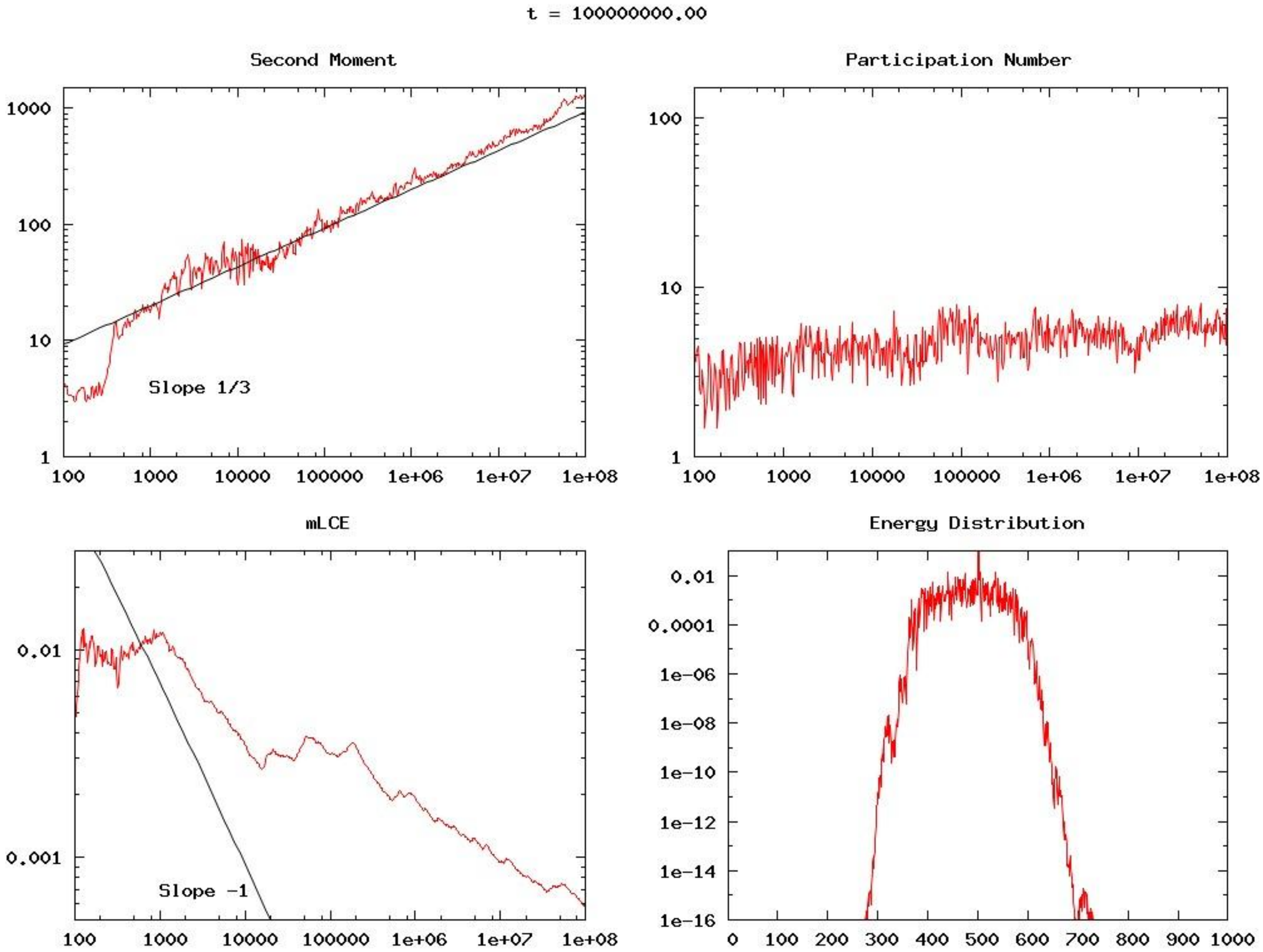
Single site excitations (E=0.4) Lyapunov exponents



KG: Weak Chaos (E=0.4)



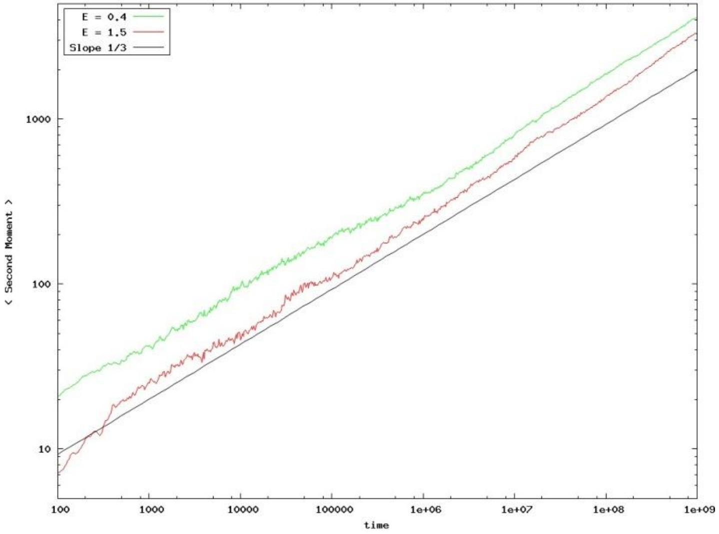
KG: Selftrapping (E=1.5)

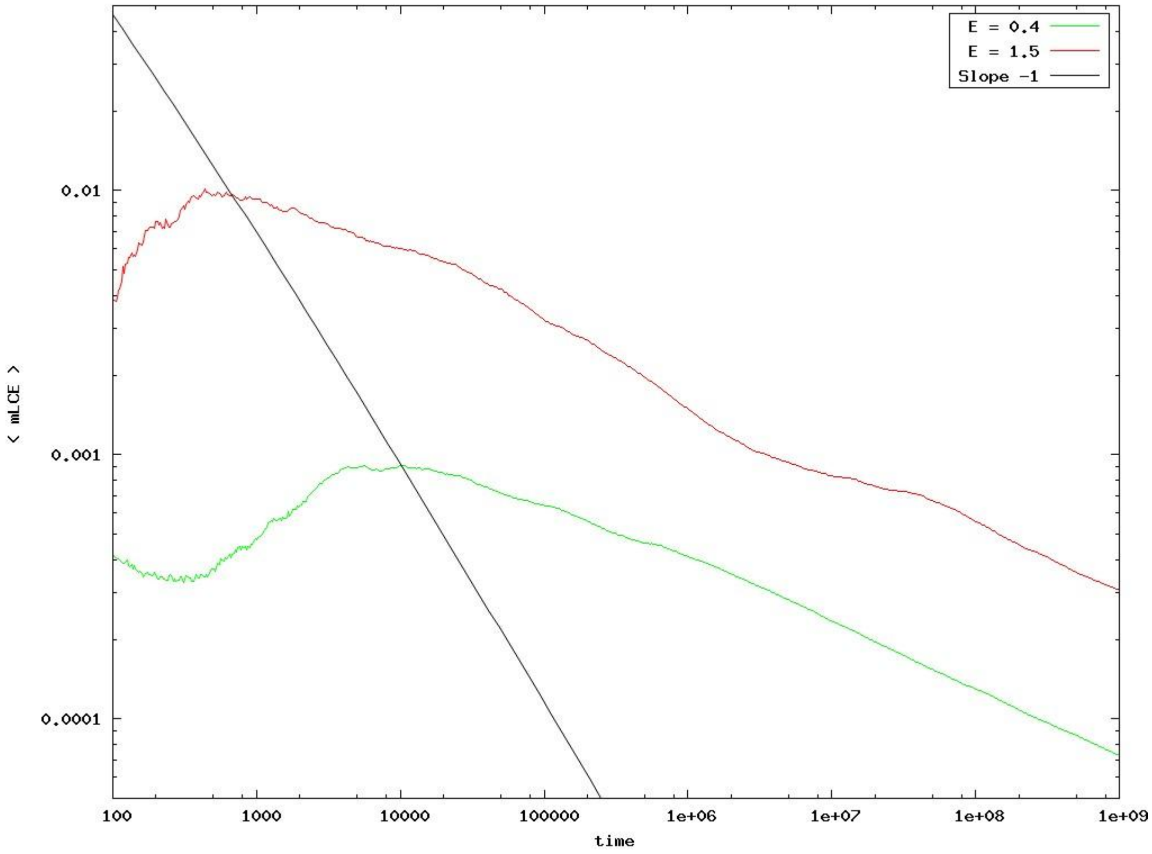


Single site excitations: Different spreading regimes

KG W = 4, E = 0.4, 1.5

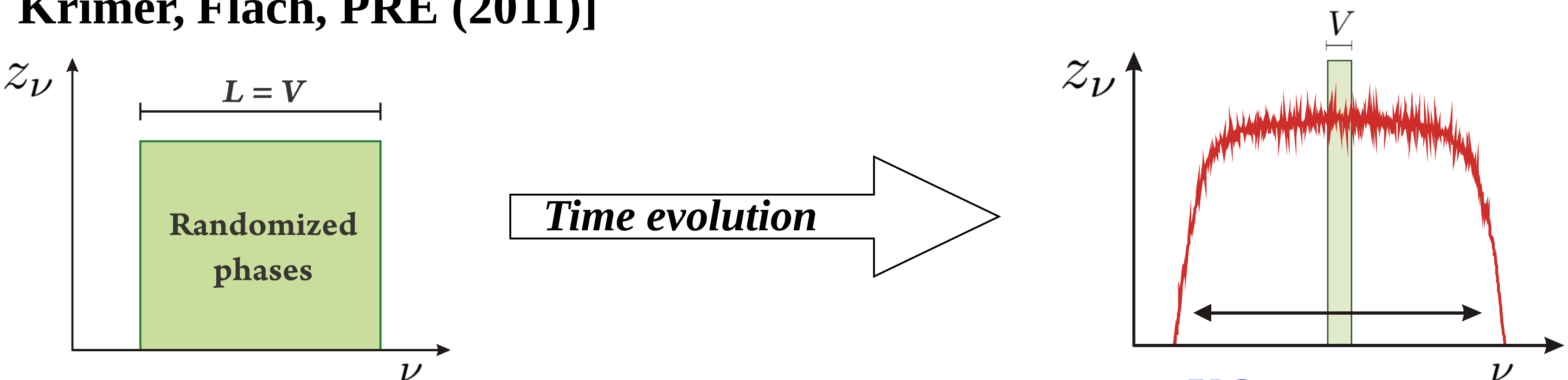
Average over 20 realizations



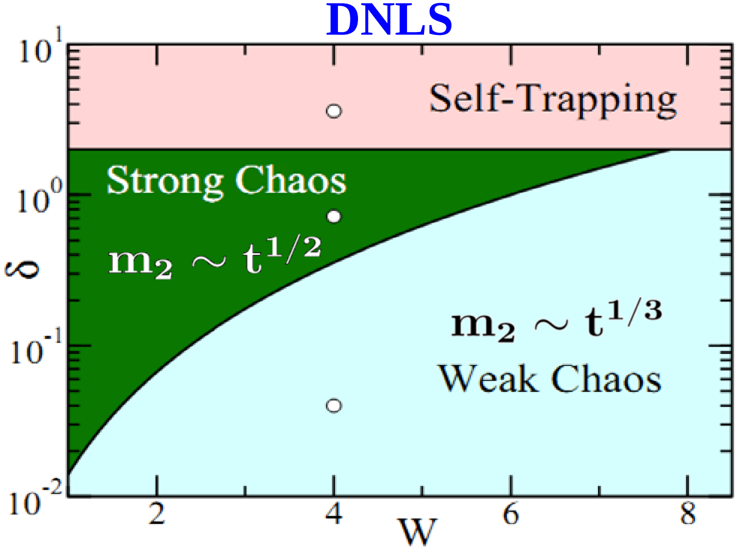


Crossover from strong to weak chaos

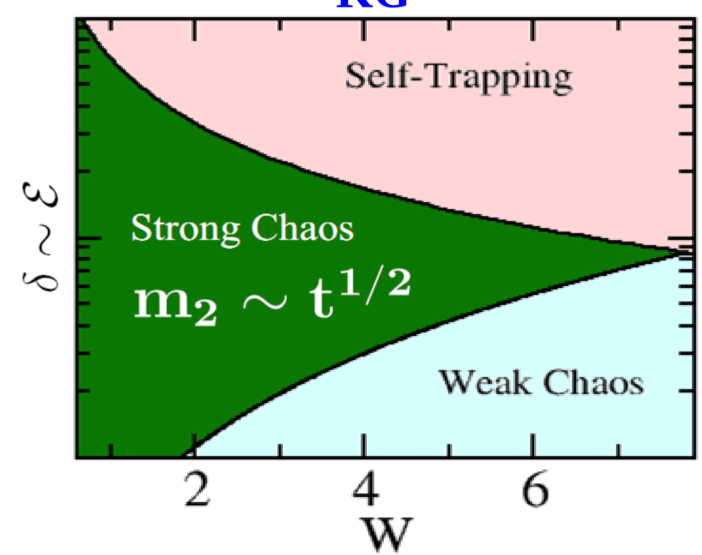
We consider compact initial wave packets of width *L=V* [Laptyeva, Bodyfelt, Krimer, Ch. S., Flach, EPL (2010) – Bodyfelt, Laptyeva, Ch. S., Krimer, Flach, PRE (2011)]



DNLS

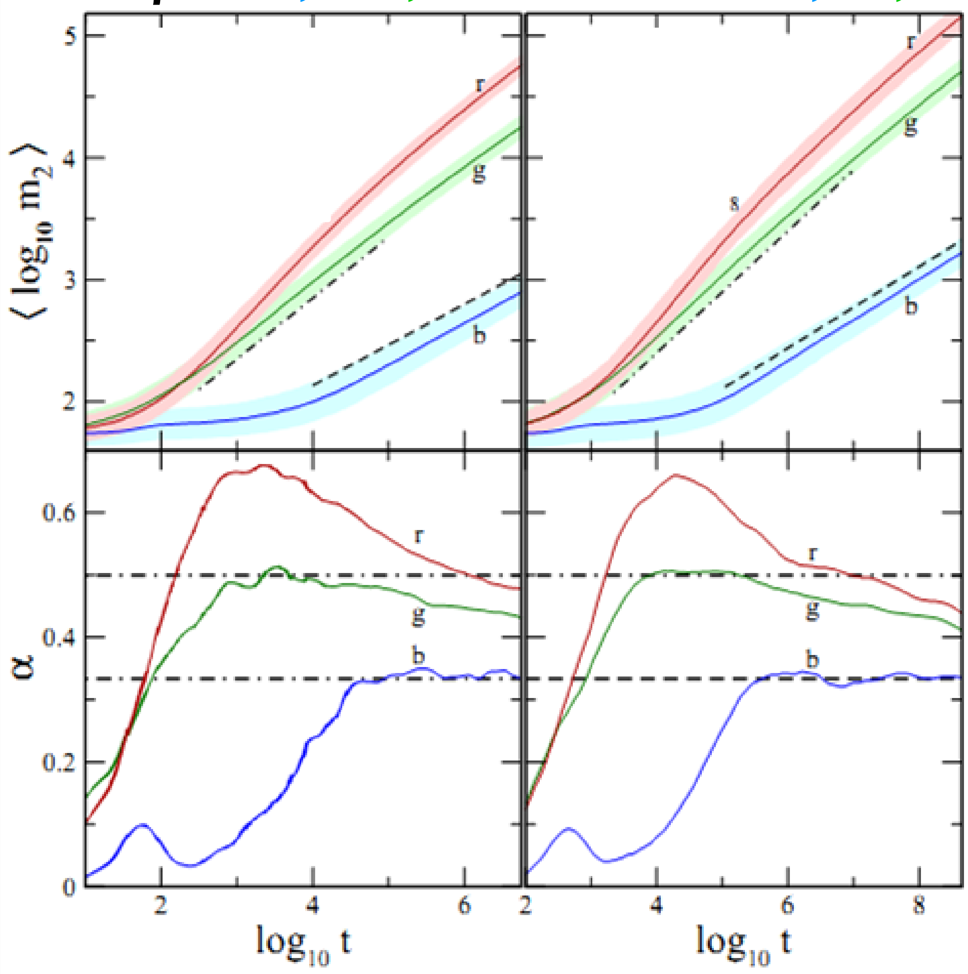


KG



Crossover from strong to weak chaos

DNLS β= 0.04, 0.72, 3.6 KG E/L= 0.01, 0.2, 0.75



W=4, L=21

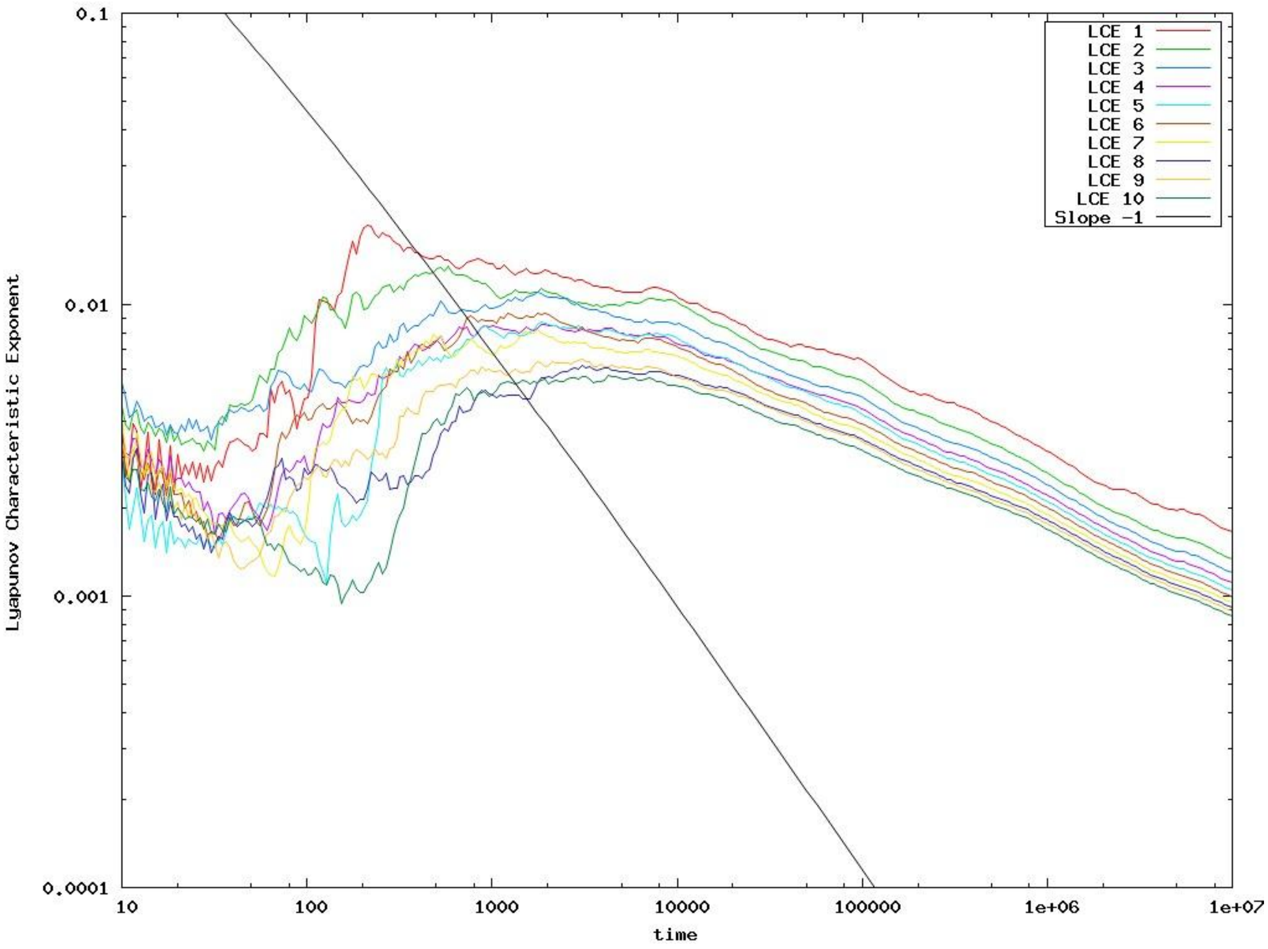
Average over 1000 realizations!

$$\alpha(\log t) = \frac{d\langle \log m_2 \rangle}{d \log t}$$

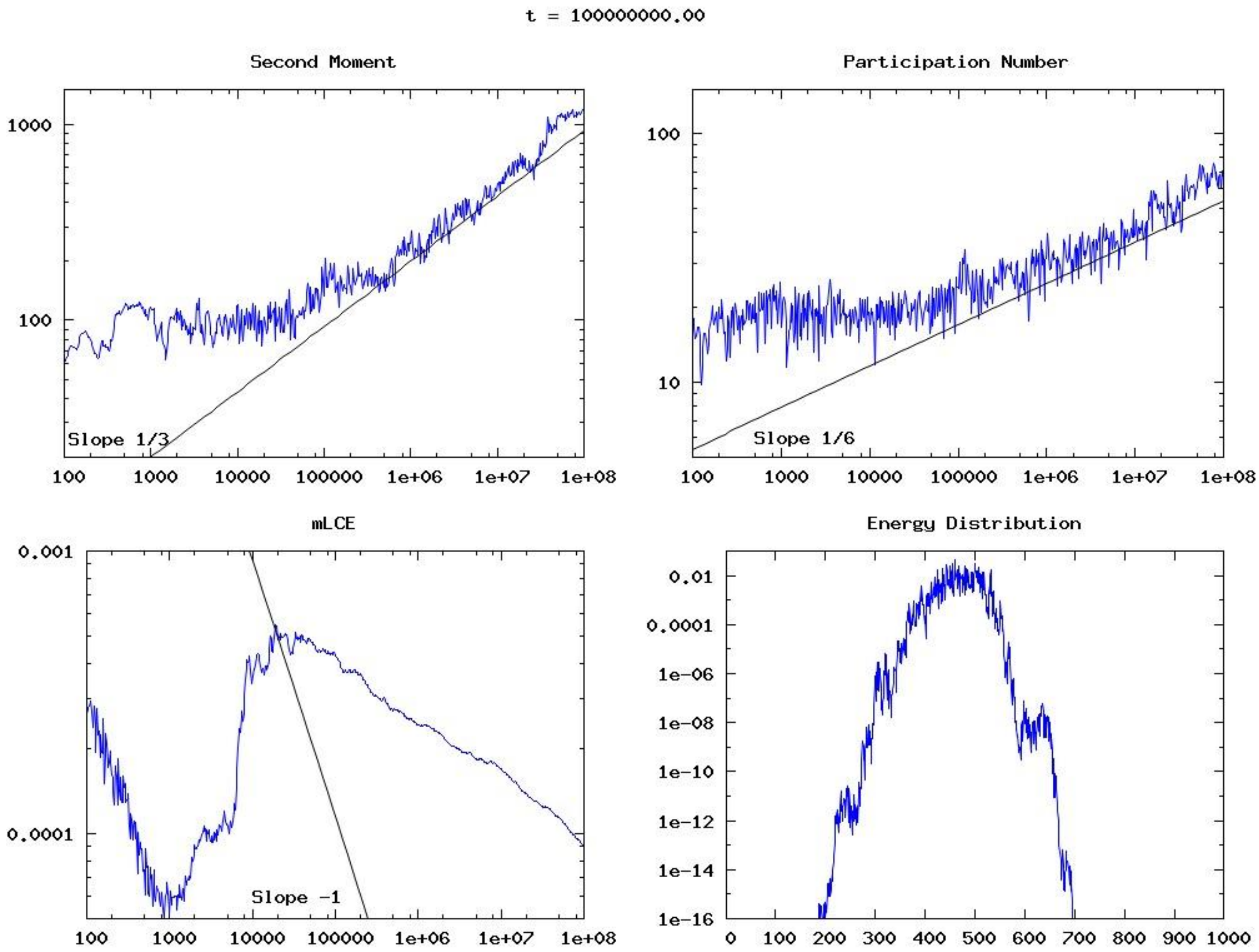
α=1/2

α=1/3

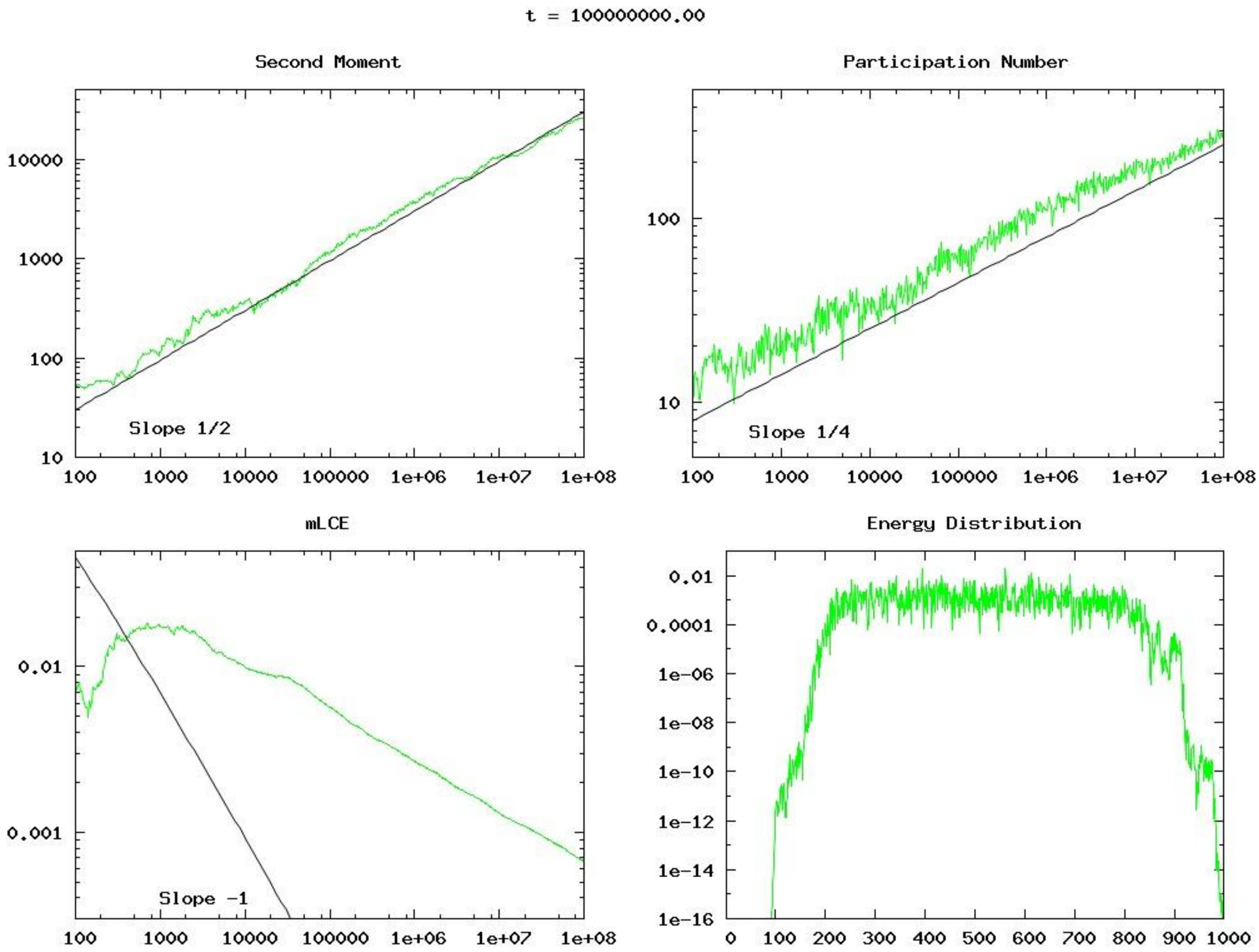
Block excitations (E/L=0.2)
Lyapunov exponents



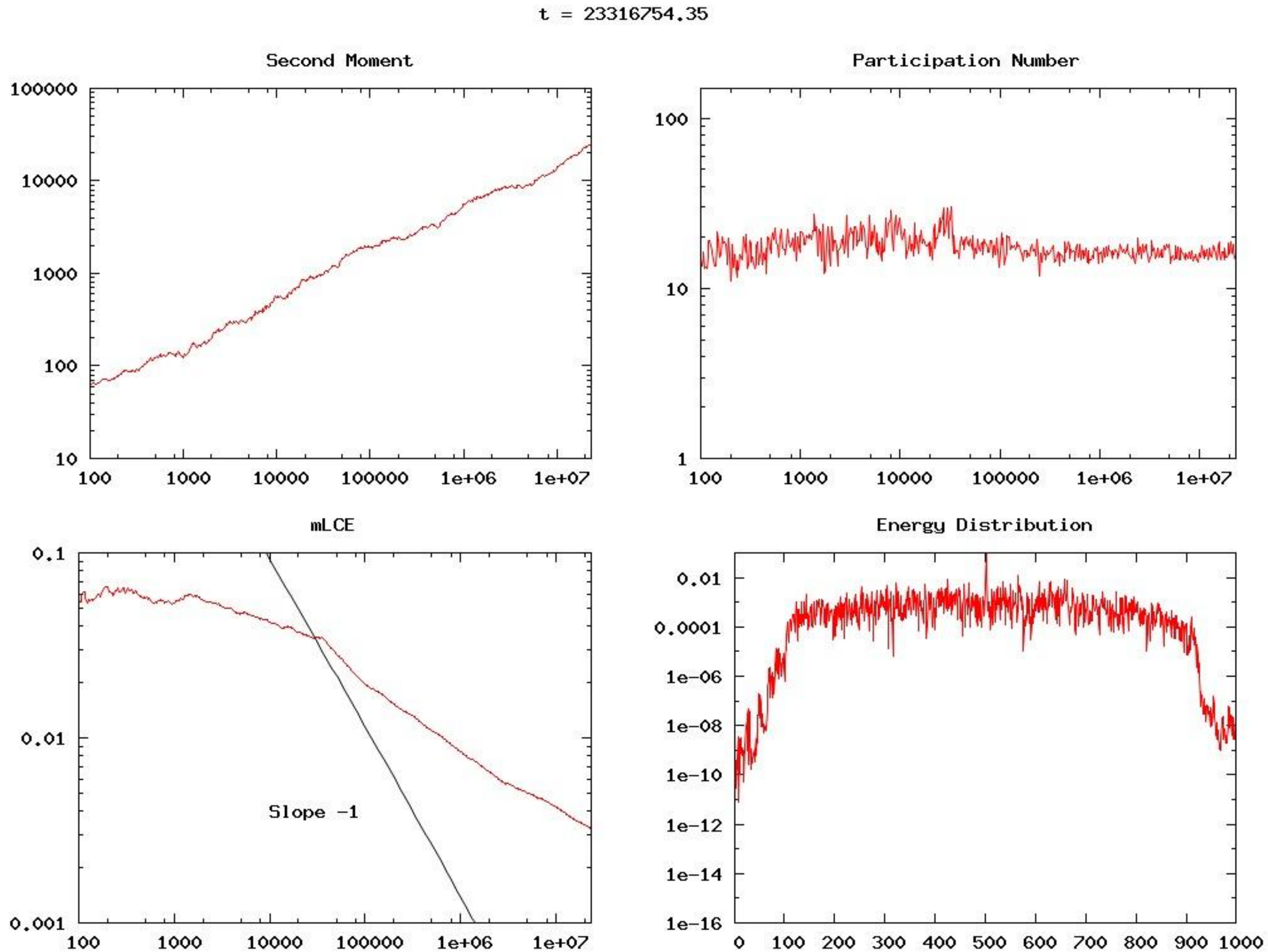
KG: Weak Chaos (E/L=0.01)



KG: Strong Chaos (E/L=0.2)



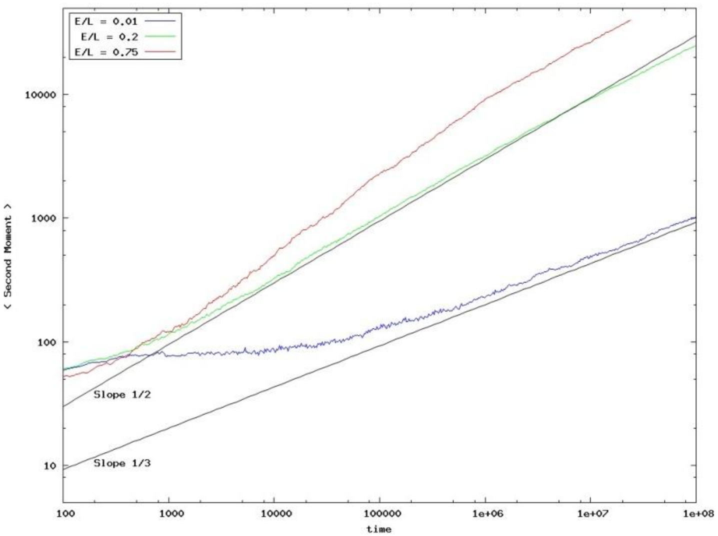
KG: Selftrapping (E/L=0.75)

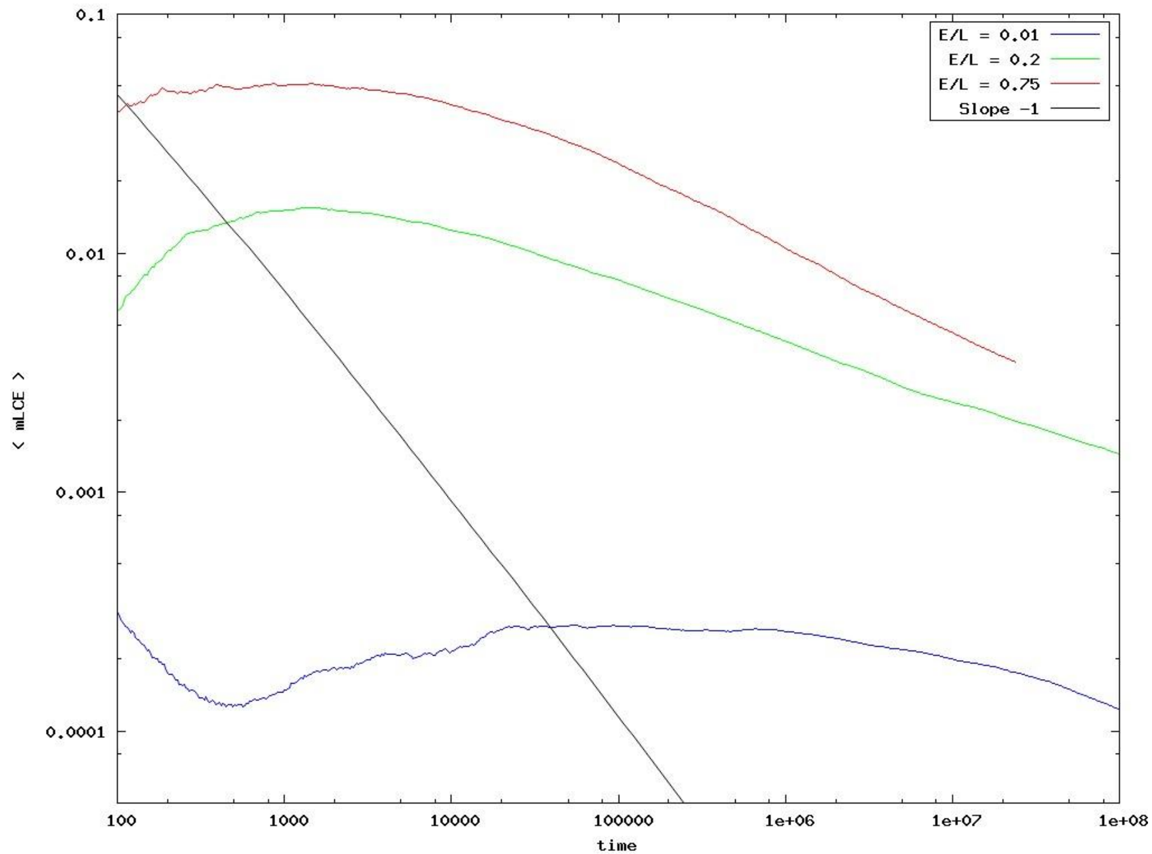


Block excitations: Different spreading regimes

W = 4, E/L = 0.01, 0.2, 0.75

Average over 20 realizations





Summary

- We predicted theoretically and verified numerically the existence of three different dynamical behaviors:
  - ✓ Weak Chaos Regime: $\delta<d$, $m_2 \sim t^{1/3}$
  - ✓ Intermediate Strong Chaos Regime: $d<\delta<\Delta$, $m_2 \sim t^{1/2} \longrightarrow m_2 \sim t^{1/3}$
  - ✓ Selftrapping Regime: $\delta>\Delta$
- Generality of results: a) Two different models: KD and DNLS, b) Predictions made for DNLS are verified for both models.
- Our results suggest that Anderson localization is eventually destroyed by the slightest amount of nonlinearity, since spreading does not show any sign of slowing down.
- Questions under investigation:
  - ✓ What is the actual chaotic nature of spreading?
  - ✓ What is the final fate of the wave packet?